# Further Rules of Differentiation

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# Rule 4 The chain rule

If $y$ is a function of $u$, which is itself a function of $x$, then

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

# To illustrate

To illustrate this rule, let us return to the function

$$y = (2x + 3)^{10}$$

in which

$$y = u^{10} \quad \text{and} \quad u = 2x + 3$$

Now

$$\frac{dy}{du} = 10u^9 = 10(2x + 3)^9$$

$$\frac{du}{dx} = 2$$

The chain rule then gives

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = 10(2x + 3)^9(2) = 20(2x + 3)^9$$

# Next..

With practice it is possible to perform the differentiation without explicitly introducing the variable $u$. To differentiate

$$y = (2x + 3)^{10}$$

we first differentiate the outer power function to get

$$10(2x + 3)^9$$

and then multiply by the derivative of the inner function, $2x + 3$, which is 2, so

$$\frac{dy}{dx} = 20(2x + 3)^9$$

## Example

Differentiate

(a) $y = (3x^2 - 5x + 2)^4$

(b) $y = \dfrac{1}{3x + 7}$

(c) $y = \sqrt{(1 + x^2)}$

## Solution

(a) The chain rule shows that to differentiate $(3x^2 - 5x + 2)^4$ we first differentiate the outer power function to get

$$4(3x^2 - 5x + 2)^3$$

and then multiply by the derivative of the inner function, $3x^2 - 5x + 2$, which is $6x - 5$. Hence if

$$y = (3x^2 - 5x + 2)^4 \quad \text{then} \quad \frac{dy}{dx} = 4(3x^2 - 5x + 2)^3(6x - 5)$$

(b) To use the chain rule to differentiate

$$y = \frac{1}{3x + 7}$$

recall that reciprocals are denoted by negative powers, so that

$$y = (3x + 7)^{-1}$$

The outer power function differentiates to get

$$-(3x + 7)^{-2}$$

and the inner function, $3x + 7$, differentiates to get 3. By the chain rule we just multiply these together to deduce that

$$\text{if} \quad y = \frac{1}{3x + 7} \quad \text{then} \quad \frac{dy}{dx} = -(3x + 7)^{-2}(3) = \frac{-3}{(3x + 7)^2}$$

(c) To use the chain rule to differentiate

$$y = \sqrt{(1 + x^2)}$$

recall that roots are denoted by fractional powers, so that

$$y = (1 + x^2)^{1/2}$$

The outer power function differentiates to get

$$\frac{1}{2}(1 + x^2)^{-1/2}$$

and the inner function, $1 + x^2$, differentiates to get $2x$. By the chain rule we just multiply these together to deduce that

$$\text{if} \quad y = \sqrt{(1 + x^2)} \quad \text{then} \quad \frac{dy}{dx} = \frac{1}{2}(1 + x^2)^{-1/2}(2x) = \frac{x}{\sqrt{(1 + x^2)}}$$

# Practice Problem

1 Differentiate

(a) $y = (3x - 4)^5$

(b) $y = (x^2 + 3x + 5)^3$

(c) $y = \dfrac{1}{2x - 3}$

(d) $y = \sqrt{(4x - 3)}$

# Rule 5 The product rule

If $y = uv$ then $\frac{dy}{dx} = u\frac{dv}{dx} \times v\frac{du}{dx}$

This rule tells you how to differentiate the product of two functions:

**multiply each function by the derivative of the other and add**

## Example

Differentiate

(a) $y = x^2(2x + 1)^3$ (b) $x\sqrt{(6x + 1)}$ (c) $y = \frac{x}{1 + x}$

## Solution

(a) The function $x^2(2x+1)^3$ involves the product of two simpler functions, namely $x^2$ and $(2x+1)^3$, which we denote by $u$ and $v$ respectively. (It does not matter which function we label $u$ and which we label $v$. The same answer is obtained if $u$ is $(2x+1)^3$ and $v$ is $x^2$. You might like to check this for yourself later.) Now if

$$u = x^2 \quad \text{and} \quad v = (2x+1)^3$$

then

$$\frac{du}{dx} = 2x \quad \text{and} \quad \frac{dv}{dx} = 6(2x+1)^2$$

where we have used the chain rule to find $dv/dx$. By the product rule,

$$\begin{aligned}\frac{dy}{dx} &= u\frac{dv}{dx} + v\frac{du}{dx} \\ &= x^2[6(2x+1)^2] + (2x+1)^3(2x)\end{aligned}$$

The first term is obtained by leaving $u$ alone and multiplying it by the derivative of $v$. Similarly, the second term is obtained by leaving $v$ alone and multiplying it by the derivative of $u$.

If desired, the final answer may be simplified by taking out a common factor of $2x(2x+1)^2$. This factor goes into the first term $3x$ times and into the second $2x+1$ times. Hence

$$\begin{aligned}\frac{dy}{dx} &= 2x(2x+1)^2[3x+(2x+1)] \\ &= 2x(2x+1)^2(5x+1)\end{aligned}$$

**(b)** The function $x\sqrt{(6x+1)}$ involves the product of the simpler functions

$$u = x \quad \text{and} \quad v = \sqrt{(6x+1)} = (6x+1)^{1/2}$$

for which

$$\frac{du}{dx} = 1 \quad \text{and} \quad \frac{dv}{dx} = \frac{1}{2}(6x+1)^{-1/2} \times 6 = 3(6x+1)^{-1/2}$$

where we have used the chain rule to find $dv/dx$. By the product rule,

$$\begin{aligned}\frac{dy}{dx} &= u\frac{dv}{dx} + v\frac{du}{dx} \\ &= x[3(6x+1)^{-1/2}] + (6x+1)^{1/2}(1) \\ &= \frac{3x}{\sqrt{(6x+1)}} + \sqrt{(6x+1)}\end{aligned}$$

If desired, this can be simplified by putting the second term over a common denominator

$$\sqrt{(6x+1)}$$

To do this we multiply the top and bottom of the second term by $\sqrt{(6x+1)}$ to get

$$\frac{6x+1}{\sqrt{(6x+1)}}$$

$\sqrt{(6x+1)} \times \sqrt{(6x+1)}$
$= 6x + 1$

Hence

$$\frac{dy}{dx} = \frac{3x + (6x+1)}{\sqrt{(6x+1)}} = \frac{9x+1}{\sqrt{(6x+1)}}$$

(c) At first sight it is hard to see how we can use the product rule to differentiate

$$\frac{x}{1+x}$$

since it appears to be the quotient and not the product of two functions. However, if we recall that reciprocals are equivalent to negative powers, we may rewrite it as

$$x(1+x)^{-1}$$

It follows that we can put

$$u = x \quad \text{and} \quad v = (1+x)^{-1}$$

which gives

$$\frac{\mathrm{d}u}{\mathrm{d}x} = 1 \quad \text{and} \quad \frac{\mathrm{d}v}{\mathrm{d}x} = -(1+x)^{-2}$$

where we have used the chain rule to find d$v$/d$x$. By the product rule

$$\frac{\mathrm{d}y}{\mathrm{d}x} = u\frac{\mathrm{d}v}{\mathrm{d}x} + v\frac{\mathrm{d}u}{\mathrm{d}x}$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = x[-(1+x)^{-2}] + (1+x)^{-1}(1)$$

$$= \frac{-x}{(1+x)^2} + \frac{1}{1+x}$$

If desired, this can be simplified by putting the second term over a common denominator

$$(1+x)^2$$

To do this we multiply the top and bottom of the second term by $1 + x$ to get

$$\frac{1+x}{(1+x)^2}$$

Hence

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{-x}{(1+x)^2} + \frac{1+x}{(1+x)^2} = \frac{-x+(1+x)}{(1+x)^2} = \frac{1}{(1+x)^2}$$

## Practice Problem

2 Differentiate

(a) $y = x(3x-1)^6$

(b) $y = x^3\sqrt{(2x+3)}$

(c) $y = \dfrac{x}{x-2}$

# Rule 6 The quotient rule

If $y = \frac{u}{v}$ then $\frac{dy}{dx} = \frac{v du/dx - u dv/dx}{v^2}$

This rule tells you how to differentiate the quotient of two functions:

**bottom times derivative of top, minus top times derivative of bottom, all over bottom squared**

## Example

Differentiate

(a) $y = \dfrac{x}{1+x}$ (b) $y = \dfrac{1+x^2}{2-x^3}$

## Solution

(a) In the quotient rule, $u$ is used as the label for the numerator and $v$ is used for the denominator, so to differentiate

$$\frac{x}{1+x}$$

we must take

$$u = x \quad \text{and} \quad v = 1 + x$$

for which

$$\frac{du}{dx} = 1 \quad \text{and} \quad \frac{dv}{dx} = 1$$

By the quotient rule

$$\frac{dy}{dx} = \frac{v\,du/dx - u\,dv/dx}{v^2}$$

$$= \frac{(1+x)(1) - x(1)}{(1+x^2)}$$

$$= \frac{1+x-x}{(1+x)^2}$$

$$= \frac{1}{(1+x)^2}$$

(b) The numerator of the algebraic fraction

$$\frac{1+x^2}{2-x^3}$$

is $1 + x^2$ and the denominator is $2 - x^3$, so we take

$$u = 1 + x^2 \quad \text{and} \quad v = 2 - x^3$$

for which

$$\frac{du}{dx} = 2x \quad \text{and} \quad \frac{dv}{dx} = -3x^2$$

By the quotient rule

$$\frac{dy}{dx} = \frac{v\,du/dx - u\,dv/dx}{v^2}$$

$$= \frac{(2-x^3)(2x) - (1+x^2)(-3x^2)}{(2-x^3)^2}$$

$$= \frac{4x - 2x^4 + 3x^2 + 3x^4}{(2-x^3)^2}$$

$$= \frac{x^4 + 3x^2 + 4x}{(2-x^3)^2}$$

## Practice Problem

**3** Differentiate

(a) $y = \dfrac{x}{x-2}$

(b) $y = \dfrac{x-1}{x+1}$

- Stop…………………!