Rules of Differentiation

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Rule 1 The constant rule

If $h(x) = cf(x)$ then $h'(x) = cf'(x)$

for any constant c.

This rule tells you how to find the derivative of a constant multiple of a function:

differentiate the function and multiply by the constant

Example

Differentiate

(a) $y = 2x^4$ (b) $y = 10x$

Jawaban

(a) To differentiate $2x^4$ we first differentiate x^4 to get $4x^3$ and then multiply by 2. Hence

$$\text{if} \quad y = 2x^4 \quad \text{then} \quad \frac{dy}{dx} = 2(4x^3) = 8x^3$$

(b) To differentiate $10x$ we first differentiate x to get 1 and then multiply by 10. Hence

$$\text{if} \quad y = 10x \quad \text{then} \quad \frac{dy}{dx} = 10(1) = 10$$

Rule 2 The sum rule

If $h(x) = f(x) + g(x)$ then $h'(x) = f'(x) + g'(x)$

This rule tells you how to find the derivative of the sum of two functions:

differentiate each function separately and add

Contoh

Example

Differentiate

(a) $y = x^2 + x^{50}$ (b) $y = x^3 + 3$

Jawaban

(a) To differentiate $x^2 + x^{50}$ we need to differentiate x^2 and x^{50} separately and to add. Now

x^2 differentiates to $2x$

and

x^{50} differentiates to $50x^{49}$

so

if $y = x^2 + x^{50}$ then $\dfrac{dy}{dx} = 2x + 50x^{49}$

(b) To differentiate $x^3 + 3$ we need to differentiate x^3 and 3 separately and to add. Now

x^3 differentiates to $3x^2$

and

3 differentiates to 0

constants differentiate to zero

so

if $y = x^3 + 3$ then $\dfrac{dy}{dx} = 3x^2 + 0 = 3x^2$

Rule 3 The difference rule

If $h(x) = f(x) - g(x)$ then $h'(x) = f'(x) - g'(x)$

This rule tells you how to find the derivative of the difference of two functions:

differentiate each function separately and subtract

Contoh

Differentiate

(a) $y = x^5 - x^2$ (b) $y = x - \frac{1}{x^2}$

Jawab

(a) To differentiate $x^5 - x^2$ we need to differentiate x^5 and x^2 separately and to subtract. Now

x^5 differentiates to $5x^4$

and

x^2 differentiates to $2x$

so

if $y = x^5 - x^2$ then $\frac{dy}{dx} = 5x^4 - 2x$

(b) To differentiate $x - \frac{1}{x^2}$ we need to differentiate x and $\frac{1}{x^2}$ separately and subtract. Now

x differentiates to 1

and

$\frac{1}{x^2}$ differentiates to $-\frac{2}{x^3}$

x^{-2} differentiates to $-2x^{-3}$

so

if $y = x - \frac{1}{x^2}$ then $\frac{dy}{dx} = 1 - \left(-\frac{2}{x^3}\right) = 1 + \frac{2}{x^3}$

Contoh

Differentiate

(a) $y = 3x^5 + 2x^3$

(b) $y = x^3 + 7x^2 - 2x + 10$

(c) $y = 2\sqrt{x} + \frac{3}{x}$

Jawab

(a) The sum rule shows that to differentiate $3x^5 + 2x^3$ we need to differentiate $3x^5$ and $2x^3$ separately and to add. By the constant rule

$3x^5$ differentiates to $3(5x^4) = 15x^4$

and

$2x^3$ differentiates to $2(3x^2) = 6x^2$

so

$$\text{if} \quad y = 3x^5 + 2x^3 \quad \text{then} \quad \frac{dy}{dx} = 15x^4 + 6x^2$$

With practice you will soon find that you can just write the derivative down in a single line of working by differentiating term by term. For the function

$$y = 3x^5 + 2x^3$$

we could just write

$$\frac{dy}{dx} = 3(5x^4) + 2(3x^2) = 15x^4 + 6x^2$$

Next

(**b**) So far we have only considered expressions comprising at most two terms. However, the sum and difference rules still apply to lengthier expressions, so we can differentiate term by term as before. For the function

$$y = x^3 + 7x^2 - 2x + 10$$

we get

$$\frac{dy}{dx} = 3x^2 + 7(2x) - 2(1) + 0 = 3x^2 + 14x - 2$$

Next

(c) To differentiate

$$y = 2\sqrt{x} + \frac{3}{x}$$

we first rewrite it using the notation of indices as

$$y = 2x^{1/2} + 3x^{-1}$$

Differentiating term by term then gives

$$\frac{dy}{dx} = 2\left(\frac{1}{2}\right)x^{-1/2} + 3(-1)x^{-2} = x^{-1/2} - 3x^{-2}$$

which can be written in the more familiar form

$$= \frac{1}{\sqrt{x}} - \frac{3}{x^2}$$

Second-Order Derivative

Whenever a function is differentiated, the thing that you end up with is itself a function. This suggests the possibility of differentiating a second time to get the 'slope of the slope function'. This is written as

$f''(x)$ read '*f* double dashed of *x*'

or

$\frac{d^2y}{dx^2}$ read 'dee two *y* by dee *x* squared'

For example, if

$$f(x) = 5x^2 - 7x + 12$$

then differentiating once gives

$$f'(x) = 10x - 7$$

and if we now differentiate $f'(x)$ we get

$$f''(x) = 10$$

The function $f'(x)$ is called the *first-order derivative* and $f''(x)$ is called the *second-order derivative*.

Contoh

Evaluate $f''(1)$ where

$$f(x) = x^7 + \frac{1}{x}$$

Solution

To find $f''(1)$ we need to differentiate

$$f(x) = x^7 + x^{-1}$$

twice and to put $x = 1$ into the end result. Differentiating once gives

$$f'(x) = 7x^6 + (-1)x^{-2} = 7x^6 - x^{-2}$$

and differentiating a second time gives

$$f''(x) = 7(6x^5) - (-2)x^{-3} = 42x^5 + 2x^{-3}$$

Next

Finally, substituting $x = 1$ into

$$f''(x) = 42x^5 + \frac{2}{x^3}$$

gives

$$f''(1) = 42 + 2 = 44$$

Latihan Soal

1 Differentiate

(a) $y = 4x^3$ (b) $y = 2/x$

2 Differentiate

(a) $y = x^5 + x$ (b) $y = x^2 + 5$

3 Differentiate

(a) $y = x^2 - x^3$ (b) $y = 50 - \frac{1}{x^3}$

4 Differentiate

(a) $y = 9x^5 + 2x^2$ (b) $y = 5x^8 - \frac{3}{x}$

(c) $y = x^2 + 6x + 3$ (d) $y = 2x^4 + 12x^3 - 4x^2 + 7x - 400$

5 Evaluate $f''(6)$ where

$f(x) = 4x^3 - 5x^2$

Aplikasi dalam Ekonomi : Marginal Function

Revenue and cost

In Chapter 2 we investigated the basic properties of the revenue function, TR. It is defined to be PQ, where P denotes the price of a good and Q denotes the quantity demanded. In practice, we usually know the demand equation, which provides a relationship between P and Q. This enables a formula for TR to be written down solely in terms of Q. For example, if

$$P = 100 - 2Q$$

then

$$\begin{aligned} \text{TR} &= PQ \\ &= (100 - 2Q)Q \\ &= 100Q - 2Q^2 \end{aligned}$$

The formula can be used to calculate the value of TR corresponding to any value of Q. Not content with this, we are also interested in the effect on TR of a change in the value of Q from some existing level. To do this we introduce the concept of marginal revenue. The ***marginal revenue***, MR, of a good is defined by

$$\text{MR} = \frac{\text{d(TR)}}{\text{d}Q}$$

Next

marginal revenue is the derivative of total revenue with respect to demand

For example, the marginal revenue function corresponding to

$$TR = 100Q - 2Q^2$$

is given by

$$\frac{d(TR)}{dQ} = 100 - 4Q$$

If the current demand is 15, say, then

$$MR = 100 - 4(15) = 40$$

Contoh 1

Example

If the demand function is

$$P = 120 - 3Q$$

find an expression for TR in terms of Q.

Find the value of MR at $Q = 10$ using

(a) differentiation

(b) the 1 unit increase approach

Jawab

Solution

$$TR = PQ = (120 - 3Q)Q = 120Q - 3Q^2$$

(a) The general expression for MR is given by

$$\frac{d(TR)}{dQ} = 120 - 6Q$$

so at $Q = 10$,

$$MR = 120 - 6 \times 10 = 60$$

(b) From the non-calculus definition we need to find the change in TR as Q increases from 10 to 11.

Putting $Q = 10$ gives $TR = 120 \times 10 - 3 \times 10^2 = 900$

Putting $Q = 11$ gives $TR = 120 \times 11 - 3 \times 11^2 = 957$

and so $MR \simeq 57$

Contoh 2

If the total revenue function of a good is given by

$$100Q - Q^2$$

write down an expression for the marginal revenue function. If the current demand is 60, estimate the change in the value of TR due to a 2 unit increase in Q.

Jawab

If

$$TR = 100Q - Q^2$$

then

$$MR = \frac{d(TR)}{dQ}$$
$$= 100 - 2Q$$

When $Q = 60$

$$MR = 100 - 2(60) = -20$$

If Q increases by 2 units, $\Delta Q = 2$ and the formula

$$\Delta(TR) \simeq MR \times \Delta Q$$

shows that the change in total revenue is approximately

$$(-20) \times 2 = -40$$

A 2 unit increase in Q therefore leads to a decrease in TR of about 40.

Latihan Soal

1 If the demand function is

$$P = 60 - Q$$

find an expression for TR in terms of Q.

(1) Differentiate TR with respect to Q to find a general expression for MR in terms of Q. Hence write down the exact value of MR at $Q = 50$.

(2) Calculate the value of TR when

(a) $Q = 50$ (b) $Q = 51$

and hence confirm that the 1 unit increase approach gives a reasonable approximation to the exact value of MR obtained in part (1).

Next

2 If the total revenue function of a good is given by

$$1000Q - 4Q^2$$

write down an expression for the marginal revenue function. If the current demand is 30, find the approximate change in the value of TR due to a

(a) 3 unit increase in Q

(b) 2 unit decrease in Q

Selesai